

We study a noise-induced bifurcation in the vicinity of the threshold by using a perturbative expansion of the order parameter, called the Poincaré-Lindstedt expansion. Each term of this series becomes divergent in the long time limit if the power spectrum of the noise does not vanish at zero frequency. These divergencies have a physical consequence : they modify the scaling of all the moments of the order parameter near the threshold and lead to a multifractal behaviour. We derive this anomalous scaling behaviour analytically by a resummation of the Poincaré-Lindstedt series and show that the usual, deterministic, scalings are recovered when the noise has a low frequency cut-off. Our analysis reconciles apparently contradictory results found in the literature.

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## I. INTRODUCTION

A random noise can strongly affect the qualitative behaviour of a nonlinear dynamical system by shifting the bifurcation threshold, by modifying the characteristic exponents or by inducing unexpected transitions [1, 2]. A straightforward approach to study the effect of noise on a bifurcation diagram would consist in analyzing the nonlinear Langevin equation that governs the system. However, the interplay between noise and nonlinearity results in subtle effects that make nonlinear stochastic differential equations hard to handle [3, 4]. Therefore, most of the works either consider the linearized stochastic system in the neighbourhood of a stable manifold (the nonlinearity is thus eliminated) or analyze Fokker-Planck type evolution equations for the Probability Distribution Function (PDF) (the noise is thus integrated out by mapping a stochastic ordinary differential equation into a deterministic partial differential equation in the phase space of the system). Any type of noise can be studied by the linearization technique, however the behaviour of the linearized dynamics is in general different from that of the real nonlinear system. On the other hand, Fokker-Planck equations fully take into account the nonlinear dynamics but are valid only for white noise (the various ‘Effective Fokker-Planck Equations’ that have been proposed to treat colored noise have a restricted range of validity). The Fokker-Planck approach can not therefore be applied to a noise with an arbitrary spectrum.

The study of a nonlinear system subject to a deterministic forcing is a classical problem in the theory of dynamical systems (including, for example, the phenomenon of parametric resonance of an oscillator driven by a periodic modulation). Many mathematical methods have been developed to analyze this question, amongst them the Poincaré-Lindstedt expansion which is a systematic perturbative calculation free of secular divergences [5, 6]. It is natural to try to adapt the Poincaré-Lindstedt expansion to the case of nonlinear system driven by a noise [7, 8]: such an expansion is versatile enough to deal with a *nonlinear* system subject to a noise having *an arbitrary spectrum*, thus allowing one to study precisely the impact of the spectral properties of the noise on a stochastic bifurcation.

The aim of the present work is to apply the Poincaré-Lindstedt technique to the study of a stochastic Hopf bifurcation and to determine the scaling exponents in the vicinity of the bifurcation threshold. We shall show that the characteristics of the noise have a strong influence on the scaling behaviour. In particular the existence of low frequencies in the noise power spectrum results in multifractality. The perturbative expansion also explains the crossover between normal scaling and anomalous scaling and allows to resolve some controversial claims in the literature.

The plan of this work is as follows: In section II, we define the model under study, carry out the Poincaré-Lindstedt expansion and find the conditions under which this expansion becomes divergent. In section III, we extract the leading term of this expansion to all orders and show that after resummation the normal scaling behaviour is replaced by multifractality. Concluding remarks are given in the last section.

## II. PERTURBATIVE ANALYSIS OF A NOISY HOPF BIFURCATION

The random dynamical system we study here has been widely discussed as a paradigm for a noise-induced bifurcation [1, 9] and is governed by the following stochastic equation :

$$\dot{x} = (\epsilon + \Delta\xi(t)) x - x^{2p+1}, \quad (1)$$

where  $p$  is a strictly positive integer and the noise  $\xi(t)$  is a Gaussian stationary random process with zero mean value and with a correlation function given by

$$\langle \xi(t)\xi(t') \rangle = \mathcal{D}(|t - t'|). \quad (2)$$

The power spectrum of the noise is the Fourier transform of the correlation function

$$\hat{\mathcal{D}}(\omega) = \int_{-\infty}^{+\infty} dt \exp(i\omega t) \langle \xi(t)\xi(0) \rangle = \int_{-\infty}^{+\infty} dt \exp(i\omega t) \mathcal{D}(|t|). \quad (3)$$

When the noise  $\xi(t)$  is white, the correlation function  $\mathcal{D}$  is a Dirac delta function and the power spectrum is a constant. We also recall that, because of the Wiener-Khinchin theorem [3], the function  $\hat{\mathcal{D}}(\omega)$  is non-negative.

Applying elementary dimensional analysis to equation (1), we obtain the following scaling relations:

$$x \sim t^{1/(2p)}, \quad \xi \sim t^{-1/2}, \quad \epsilon \sim \Delta^2 \sim t^{-1}. \quad (4)$$

The dimension of the noise  $\xi$  is so chosen as to render the power spectrum  $\hat{\mathcal{D}}(\omega)$  a dimensionless function.

### A. The white noise case

When  $\xi(t)$  is a Gaussian white noise, the stationary solution of the Fokker-Planck equation corresponding to equation (1) is given by

$$P_{\text{stat}}(x) = \frac{2p}{\Gamma(\alpha)(p\Delta^2)^\alpha} x^{2p\alpha-1} \exp\left(-\frac{x^{2p}}{p\Delta^2}\right) \quad \text{with } \alpha = \frac{\epsilon}{p\Delta^2}, \quad (5)$$

and where  $\Gamma$  represents the Euler Gamma-function. The bifurcation threshold is given by  $\epsilon = 0$ ; for  $\epsilon < 0$ , the solution (5) is not normalizable : the stationary PDF is the Dirac distribution  $\delta(x)$  localized at the absorbing fixed point  $x = 0$ . For  $\epsilon > 0$ , the solution (5) is normalizable and is the extended stationary PDF. In this case, the moments of  $x$  are given by

$$\langle x^{2n} \rangle = \frac{\Gamma(\alpha + n/p)}{\Gamma(\alpha)} (p\Delta^2)^{n/p}. \quad (6)$$

In the vicinity of the threshold,  $\epsilon$  is small and we find that the moments scale linearly with  $\epsilon$ , *i.e.*,

$$\langle x^{2n} \rangle \simeq \epsilon (p\Delta^2)^{n/p-1} \Gamma(n/p). \quad (7)$$

In the following, we shall find the characteristics of the noise for which such an anomalous scaling is valid.

### B. The threshold shift

The presence of noise can modify the bifurcation threshold which is given by  $\epsilon = 0$  in the deterministic case. For the stochastic differential equation (1) subject to an arbitrary noise  $\xi(t)$ , the critical value  $\epsilon_c(\Delta)$  that separates a localised PDF from an extended PDF, is determined by the vanishing of the Lyapunov exponent associated with the fixed point  $x = 0$ . We linearize equation (1) around  $x = 0$ ,

$$\frac{d\delta x}{dt} = (\epsilon + \Delta\xi(t)) \delta x, \quad (8)$$

and find the Lyapunov exponent  $\Lambda$  to be

$$\Lambda = \lim_{t \rightarrow \infty} \frac{d}{dt} \langle \ln(\delta x) \rangle = \left\langle \frac{\frac{d\delta x}{dt}}{\delta x} \right\rangle = \epsilon. \quad (9)$$

This result is valid regardless of the nature of the noise  $\xi$  (we have only used the fact that the mean value of  $\xi$  vanishes). The Lyapunov exponent vanishes when  $\epsilon = 0$ . Thus, for the first order equation (1), there is no threshold shift due to the presence of noise and we always have  $\epsilon_c(\Delta) = 0$ .

In order to simplify the general treatment, we shall study the case of a cubic nonlinearity, *i.e.*, we take  $p = 1$  in equation (1). The Poincaré-Lindstedt method [5, 6, 7, 8] consists in writing two expansions : one for the solution  $x(t)$  of equation (1) and another for the deviation from threshold  $(\epsilon - \epsilon_c(\Delta))$ , both as power series of a formal parameter  $\lambda$ ,

$$x(t) = \lambda x_1(t) + \lambda^2 x_2(t) + \lambda^3 x_3(t) + \lambda^4 x_4(t) + \dots, \quad (10)$$

$$\epsilon - \epsilon_c(\Delta) = \lambda \epsilon_1 + \lambda^2 \epsilon_2 + \lambda^3 \epsilon_3 + \lambda^4 \epsilon_4 + \dots. \quad (11)$$

By substituting the expansions (10 and 11) in equation (1) we observe that  $x(t)$  and  $(\epsilon - \epsilon_c(\Delta))$  are, respectively, odd and even in  $\lambda$  (because equation (1) is antisymmetric under  $x \rightarrow -x$ ). The Poincaré-Lindstedt expansion thus reduces to (using the fact that  $\epsilon_c(\Delta) = 0$ )

$$x(t) = \lambda (x_1(t) + \lambda^2 x_3(t) + \lambda^4 x_5(t) + \dots), \quad (12)$$

$$\epsilon = \lambda^2 (\epsilon_2 + \lambda^2 \epsilon_4 + \lambda^4 \epsilon_6 + \dots). \quad (13)$$

Substituting these formal expansions in equation (1) and identifying the terms order by order in  $\lambda$ , we obtain a hierarchy of first order differential equations for the functions  $x_{2i+1}(t)$

$$\mathcal{L}x_{2i+1}(t) = \mathcal{P}(x_1, x_3, \dots, x_{2i-1}, \epsilon_2, \dots, \epsilon_{2i}), \quad (14)$$

where  $\mathcal{P}$  is a polynomial function and  $\mathcal{L}$  a linear differential operator :

$$\mathcal{L} = \frac{d}{dt} - \Delta \xi(t). \quad (15)$$

The hierarchy (14) is solved with the initial conditions

$$x_1(0) = 1, \text{ and } x_{2i+1}(0) = 0 \text{ for all } i \geq 1. \quad (16)$$

These conditions imply that  $x(0) = \lambda$  : the formal parameter  $\lambda$  is simply equal to the value of  $x$  at time  $t = 0$  and, therefore, has the dimensions

$$\lambda \sim t^{1/2}. \quad (17)$$

Besides, the numbers  $\epsilon_i$  appear as parameters in equation (14) and are determined recursively thanks to the solvability condition that we now derive. Let us call  $y_1(t)$  the solution of the adjoint equation  $\mathcal{L}^\dagger y_1 = 0$ , which is given by

$$y_1(t) = \exp\left(-\Delta \int_0^t \xi(u) du\right). \quad (18)$$

Multiplying equation (1) by the function  $y_1$  and taking average values, we obtain

$$\langle y_1 \mathcal{L}x_1 \rangle = \langle \epsilon y_1 x_1 - y_1 x_1^3 \rangle. \quad (19)$$

Integrating the left hand side of this equation by parts and taking into account the fact that  $y_1$  is in the kernel of the adjoint operator  $\mathcal{L}^\dagger$ , we derive the following relation

$$\epsilon = \frac{\langle y_1 x_1^3 \rangle}{\langle y_1 x_1 \rangle}. \quad (20)$$

By virtue of this solvability condition, the hierarchy of equations (14) is defined without ambiguity. The functions  $x_i$  and the parameters  $\epsilon_i$  are determined recursively in a unique manner using the initial conditions (16).

Eliminating  $\lambda$  from equations (12) and (13) leads to the required expansion of  $x(t)$  in terms of  $\epsilon$ .

#### D. Calculation of the first terms in the expansion

We now calculate the first terms of the Poincaré-Lindstedt expansion by applying the procedure described above. It will be useful to introduce the following auxiliary random variable  $B_t$  :

$$B_t = \int_0^t \xi(u) du. \quad (21)$$

Because  $\xi$  is taken to be a Gaussian random process,  $B_t$  is also Gaussian. The lowest order terms in the Poincaré-Lindstedt expansion are then given by

$$x_1(t) = \exp(\Delta B_t), \quad (22)$$

$$x_3(t) = x_1(t) \left( \epsilon_2 t - \int_0^t x_1^2(u) du \right), \quad (23)$$

and the parameters  $\epsilon_2, \epsilon_4$  are given by

$$\epsilon_2 = \langle x_1^2(t) \rangle = \langle \exp(2\Delta B_t) \rangle, \quad (24)$$

$$\epsilon_4 = 3\langle x_1 x_3 \rangle - \epsilon_2 \left\langle \frac{x_3}{x_1} \right\rangle, \quad (25)$$

where the expectation value  $\langle . \rangle$  is taken over all the possible histories between times 0 and  $t$ .

### E. Behaviour of the moments

We now determine the behaviour of the moments of  $x$ , such as  $\langle x^{2n} \rangle$ , in the vicinity of the threshold, *i.e.*, when  $\epsilon \rightarrow 0$ . In the Poincaré-Lindstedt expansion, this corresponds to the formal parameter  $\lambda$  converging to 0. In this limit, we obtain

$$\langle x^{2n} \rangle = \lambda^{2n} (\langle x_1^{2n} \rangle + 2n\lambda^2 \langle x_1^{2n-1} x_3 \rangle + \dots) \simeq \lambda^{2n} \langle \exp(2n\Delta B_t) \rangle = \lambda^{2n} \exp(2n^2 \Delta^2 \langle B_t^2 \rangle), \quad (26)$$

$$\epsilon = \lambda^2 (\epsilon_2 + \lambda \epsilon_4 + \dots) \simeq \lambda^2 \langle \exp(2\Delta B_t) \rangle = \lambda^2 \exp(2\Delta^2 \langle B_t^2 \rangle). \quad (27)$$

The last equality is derived by using the fact that  $B_t$  is a Gaussian random process. Eliminating  $\lambda$  from equations (26) and (27), we obtain

$$\langle x^{2n} \rangle \simeq \epsilon^n \exp(2(n^2 - n)\Delta^2 \langle B_t^2 \rangle). \quad (28)$$

This equation predicts a normal scaling behaviour identical to that of the deterministic case. In fact, this scaling is merely a logical consequence of the formal structure of the Poincaré-Lindstedt expansion : indeed, we have  $\langle x^{2n} \rangle \sim \lambda^{2n}$  and  $\epsilon \sim \lambda^2$  and therefore  $\langle x^{2n} \rangle \sim \epsilon^n$ . However, the proportionality factor between  $\langle x^{2n} \rangle$  and  $\epsilon^n$  in equation (28) can be divergent when  $t \rightarrow \infty$ . Such a divergence can change the scaling behaviour of the moments in the large time limit.

### F. Importance of low frequencies in the noise spectrum

In order to determine the multiplicative factor of  $\epsilon^n$  in equation (28), we must calculate the variance of the random variable  $B_t$  :

$$\langle B_t^2 \rangle = \int_0^t \int_0^t \langle \xi(u)\xi(v) \rangle dudv = \int_0^t \int_0^t \mathcal{D}(|u-v|) dudv = \int_{-\infty}^{+\infty} \frac{1 - \cos \omega t}{\omega^2} \frac{\hat{\mathcal{D}}(\omega) d\omega}{\pi}. \quad (29)$$

The last integral is well defined at  $\omega = 0$  (the time  $t$  introduces an effective low frequency cut-off for  $\omega \sim 1/t$ ). The behaviour of  $\langle B_t^2 \rangle$  for  $t \rightarrow \infty$  depends on the behaviour of  $\hat{\mathcal{D}}(\omega)$  at  $\omega \rightarrow 0$ . The following two cases must be distinguished :

(i) The spectrum of the noise vanishes at low frequencies, *i.e.*,  $\mathcal{D}(0) = 0$ . Because  $\hat{\mathcal{D}}(\omega)$  is an even function of  $\omega$ , we suppose that  $\hat{\mathcal{D}}(\omega) \sim \omega^2$  for  $\omega \sim 0$  (we discard non-analytic behaviour of the power spectrum at the origin. Such non-analyticity would correspond to long tails in the correlation function of the noise).

(ii) The power spectrum of the noise is finite at  $\omega = 0$ , *i.e.*,  $\mathcal{D}(0) > 0$ .

In case (i), the long time limit of equation (29) is readily derived and we obtain (by using the Riemann-Lebesgue lemma)

$$\langle B_t^2 \rangle \rightarrow \int_{-\infty}^{+\infty} \frac{\hat{\mathcal{D}}(\omega) d\omega}{\pi \omega^2} \quad \text{when } t \rightarrow \infty. \quad (30)$$

The variance of  $B_t$  has a *finite* limit at large times and, therefore, the prefactor of  $\epsilon^n$  in equation (28) converges to a finite number when  $t \rightarrow \infty$ , for all  $n \geq 1$ . Thus, the Poincaré-Lindstedt expansion, used at the first order, leads

to a well-defined asymptotic behaviour for  $\langle x^{2n} \rangle$  and allows us to recover the 'normal' scaling behaviour which was predicted in [7] for the random frequency oscillator. Higher order terms in the expansion do not affect the scaling exponents and modify the prefactors only. The next order term was studied in [8].

In case (ii), the integral on the right hand side of equation (29) diverges when  $t \rightarrow \infty$  and its leading behaviour is

$$\langle B_t^2 \rangle = \frac{t}{\pi} \int_{-\infty}^{+\infty} \frac{1 - \cos u}{u^2} \hat{\mathcal{D}}\left(\frac{u}{t}\right) du \rightarrow \hat{\mathcal{D}}(0)t. \quad (31)$$

The variance of  $B_t$  grows linearly with time in the long time limit. Thus, for  $n \geq 2$ , the coefficient of  $\epsilon^n$  in equation (28) grows exponentially with time. Such a divergence of a prefactor is an indication of anomalous scaling. This anomalous scaling was not found in [8] where the authors analyzed only the first few terms of the Poincaré-Lindstedt expansion and supposed, in order to avoid divergencies, that the noise spectrum has a low frequency cut-off  $\omega_<$  (*i.e.*,  $\hat{\mathcal{D}}(\omega) = 0$  for  $\omega \leq \omega_<$ ). We will show in the next section that even when low frequencies are present, the Poincaré-Lindstedt expansion can be used to extract sound results and to predict anomalous scaling.

### III. RESUMMATION, ANOMALOUS SCALING AND INTERMITTENCY

When the power spectrum at  $\omega = 0$  is finite (*i.e.*,  $\hat{\mathcal{D}}(0) > 0$ ), all the coefficients of the Poincaré-Lindstedt series (12,13) blow up when time goes to infinity. This divergence, that appears even for the lowest order, seemingly implies that the expansion breaks down when  $t \rightarrow \infty$ . We now show that it is still possible to use the Poincaré-Lindstedt expansion and to extract from it by resummation the multifractal behaviour found in [12].

#### A. Resummation of the Poincaré-Lindstedt series

Without loss of generality, we suppose in the sequel that  $\hat{\mathcal{D}}(0) = 1$ ; this amounts simply to redefining  $\Delta$  as  $\Delta/\sqrt{\hat{\mathcal{D}}(0)}$ . We first analyse the lowest order terms in the Poincaré-Lindstedt expansion (13,24, 25) of  $\epsilon$ . If we retain only the most divergent contribution when  $t \rightarrow \infty$ , we obtain

$$\epsilon_2 \simeq \exp(2\Delta^2 t), \quad \epsilon_4 \simeq -\frac{\exp(8\Delta^2 t)}{2\Delta^2}, \quad \epsilon_6 \simeq \frac{3}{8} \frac{\exp(18\Delta^2 t)}{(2\Delta^2)^2}. \quad (32)$$

We now investigate the general structure of the Poincaré-Lindstedt series (13), retaining for each order only the most divergent term. From dimensional analysis (equations (4) and (17)), the dimensionless variable  $\epsilon/\Delta^2$  can be written in terms of the dimensionless expansion parameter  $\lambda/\Delta$  as follows

$$\frac{\epsilon}{\Delta^2} = \sum_{k=1}^{\infty} (-1)^{k-1} a_k \left( \frac{\lambda^2}{\Delta^2} \right)^k \exp(2k^2 \Delta^2 t); \quad (33)$$

using equation (32), we find  $a_1 = 1, a_2 = 1/2, a_3 = 3/32$ . In fact, the general formula for the  $k$ -th coefficient  $a_k$  can be obtained recursively using equation (14)

$$a_k = \frac{k}{4^{k-1}(k-1)!}. \quad (34)$$

We emphasize that the series (33) is divergent: its radius of convergence is strictly 0. However, it is possible to make a resummation of this divergent expansion by using the following representation for the exponential term

$$\exp(k^2) = \int_{-\infty}^{+\infty} \frac{du}{\sqrt{\pi}} \exp(-u^2 + 2ku). \quad (35)$$

Inserting this identity in equation (33), we obtain

$$\frac{\epsilon}{\Delta^2} = \int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \sum_{k=1}^{\infty} (-1)^{k-1} a_k \left( \frac{\lambda^2}{\Delta^2} \right)^k \exp(2k\Delta\sqrt{t}u). \quad (36)$$

The series under the integral sign has generically a non-zero radius of convergence that depends on the coefficients  $a_k$ . Defining

$$F(z) = \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k, \quad (37)$$

and substituting in this equation the expression (34) for the  $a_k$ 's, we obtain

$$F(4z) = 4z(1-z) \exp(-z). \quad (38)$$

Equation (36) can now be written as follows

$$\frac{\epsilon}{\Delta^2} = \int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) F\left(\frac{\lambda^2}{\Delta^2} \exp(2\Delta\sqrt{t}u)\right). \quad (39)$$

A resummation can be performed along the same lines for the mean value of  $x^{2n}$  : starting from the Poincaré-Lindstedt expansion (12), we find that

$$\frac{\langle x^{2n} \rangle}{\Delta^{2n}} = \sum_{k=0}^{\infty} (-1)^k b_k^{(n)} \left(\frac{\lambda}{\Delta}\right)^{2(n+k)} \exp(2(n+k)^2 \Delta^2 t). \quad (40)$$

Using equation (35) this divergent series is transformed as

$$\frac{\langle x^{2n} \rangle}{\Delta^{2n}} = \int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) G_n\left(\frac{\lambda^2}{\Delta^2} \exp(2\Delta\sqrt{t}u)\right), \quad (41)$$

where we have introduced the new function

$$G_n(z) = \sum_{k=0}^{\infty} (-1)^k b_k^{(n)} z^{n+k}. \quad (42)$$

To summarize, we have resummed the Poincaré-Lindstedt series and obtained the following relations :

$$\frac{\epsilon}{\Delta^2} = \mathcal{F}\left(\frac{\lambda^2}{\Delta^2}, \Delta\sqrt{t}\right) \quad \text{where } \mathcal{F}(z, w) \sim z \quad \text{when } z \rightarrow 0 \text{ and } w \text{ is finite}, \quad (43)$$

$$\frac{\langle x^{2n} \rangle}{\Delta^{2n}} = \mathcal{G}_n\left(\frac{\lambda^2}{\Delta^2}, \Delta\sqrt{t}\right) \quad \text{where } \mathcal{G}_n(z, w) \sim z^n \quad \text{when } z \rightarrow 0 \text{ and } w \text{ is finite}. \quad (44)$$

The scaling behaviour of  $\langle x^{2n} \rangle$  as a function of  $\epsilon$  in the vicinity of the bifurcation threshold is obtained by eliminating  $\lambda$  between these two equations.

## B. Anomalous scaling

The functions  $\mathcal{F}$  and  $\mathcal{G}_n$  that appear in equations (43 and 44) have a singular behaviour when  $\lambda \rightarrow 0$  and  $t \rightarrow \infty$  : these two limits *do not commute*. However, thanks to the expressions given in equations (43, 44), we can disentangle these two limits.

If we take  $\lambda \rightarrow 0$  for a large but fixed value of  $t$ , we find from equations (43) and (44) that  $\langle x^{2n} \rangle \sim \epsilon^n$ . Normal scaling is therefore recovered, in agreement with equation (28).

However, taking the limit  $t \rightarrow \infty$  first and keeping the value of  $\lambda$  fixed and small, we obtain from equation (39),

$$\frac{\epsilon}{\Delta^2} = \int_{-\infty}^{+\infty} \frac{dv}{2\Delta\sqrt{2\pi t}} \exp\left(-\frac{v^2}{8\Delta^2 t}\right) F\left(\frac{\lambda^2}{\Delta^2} \exp(v)\right) \simeq \int_{-\infty}^{+\infty} \frac{dv}{2\Delta\sqrt{2\pi t}} F\left(\frac{\lambda^2}{\Delta^2} \exp(v)\right) = \frac{\int_{-\infty}^{+\infty} dv F(\exp(v))}{2\Delta\sqrt{2\pi t}}, \quad (45)$$

where the last equality is obtained by the translation  $v \rightarrow v - \log(\lambda^2/\Delta^2)$ . Using the explicit expression (38) for  $F(z)$  we verify that the integral on the right-hand side of equation (45) is a strictly positive real number, *i.e.*, it is neither zero nor infinite. We have thus shown that

$$\frac{\epsilon}{\Delta^2} \simeq \frac{e_0}{\Delta\sqrt{2\pi t}}, \quad (46)$$

where  $e_0 > 0$ . In a similar manner, using equation (41), we find that the asymptotic behaviour of the  $2n$ -th moment of  $x$  is given by

$$\frac{\langle x^{2n} \rangle}{\Delta^{2n}} \simeq \frac{c_n}{\Delta \sqrt{2\pi t}}. \quad (47)$$

Eliminating  $t$  from equations (46) and (47) provides us the scaling behaviour of the moments of  $x$  in the vicinity of the bifurcation threshold

$$\langle x^{2n} \rangle \simeq C_n \epsilon \Delta^{2n-2}. \quad (48)$$

In the vicinity of  $\epsilon = 0$  all the moments scale linearly with  $\epsilon$ . This equation generalizes the calculation of the white noise case (7) to an *arbitrary* noise with non-vanishing zero-frequency power spectrum. We have thus shown, using the Poincaré-Lindstedt expansion, that the low frequency of the noise strongly affect the scaling of the order parameter in the vicinity of the threshold of a stochastic bifurcation and induces a multifractal behaviour; a qualitative explanation of this effect was given in [12].

### C. Discussion of the random frequency oscillator

The above analysis can be applied to more general random dynamical systems such as the parametrically driven damped anharmonic oscillator that naturally appears in the study of many instabilities [13]. Such a system is described by the following equation :

$$m\ddot{x} + m\gamma\dot{x} = (\epsilon + \Delta\xi(t))x - x^3, \quad (49)$$

where  $\epsilon$  is the control parameter and the modulation  $\xi(t)$  is of arbitrary dynamics and statistics: it can be a periodic function or a random noise. For small driving amplitudes  $\Delta$ , Lücke and Schank [7] have performed a Poincaré-Lindstedt expansion, and obtained an expression for the threshold  $\epsilon_c(\Delta)$  (at first order in  $\Delta$ ). Their result has been verified both numerically and experimentally and is also in agreement with the exact result obtained for the Gaussian white noise (in this case a closed formula is available for  $\epsilon_c(\Delta)$  for arbitrary values of  $\Delta$ ). Another result obtained in [7, 8] is the scaling of the moments near the threshold,

$$\langle x^{2n} \rangle = s_n (\epsilon - \epsilon_c(\Delta))^n + \mathcal{O}((\epsilon - \epsilon_c)^{n+1}), \quad (50)$$

where the constant  $s_n$  depends on  $\xi(t)$  and on  $\Delta$ . The moments have a *normal scaling* behaviour :  $\langle x^{2n} \rangle$  scales as  $\langle x^2 \rangle^n$ . The bifurcation scaling exponent is equal to 1/2 and is the same as that of a deterministic Hopf bifurcation. However, this expression does not agree with the results for random iterated maps, for the random parametric oscillator and with recent studies on On-Off intermittency [10, 11, 12]. These works predict that the variable  $x$  is intermittent and that the moments of  $x$  exhibit *anomalous scaling*,

$$\langle x^{2n} \rangle \simeq \kappa_n (\epsilon - \epsilon_c) \quad \text{for all } n > 0, \quad (51)$$

*i.e.*, all the moments grow linearly with the distance from threshold. This multifractal behaviour, confirmed by numerical simulations for a Gaussian white noise, was derived using effective Fokker-Planck equations.

The origin of the contradiction between equations (50) and (51) lies in the divergences that appear in the Poincaré-Lindstedt expansion. This fact, identified in [8], implies that the results of [7] are valid only for noises that do not have low frequencies. We have shown by studying a model technically simpler than equation (49), that the perturbative Poincaré-Lindstedt expansion can be used for any kind of noise by resumming the divergent terms to all orders. This resummation allows to recover the scaling exponents for all the cases, and describes the crossover between the scalings given in equations (50) and (51).

## IV. CONCLUSION

In this work, we have shown that the Poincaré-Lindstedt expansion, a classical perturbative technique extensively used in the field of nonlinear dynamical systems, can be successfully adapted to analyze a stochastic model that plays the role of a paradigm for noise-induced bifurcations. These perturbative expansions for stochastic dynamics were studied in [7], but, due to the presence of divergent terms, were applied only to random noises with vanishing power spectrum at low frequencies (e.g., noises with a low frequency cut-off) [8]. However, we have shown here that, by a

resummation of the divergent terms, the Poincaré-Lindstedt method can be used for *any type of noise*. Moreover, the divergences that appear in the Poincaré-Lindstedt expansion are not mathematical artefacts but have genuine physical consequences : the presence of low frequencies in the noise spectrum (that leads to these divergences) modifies the scaling behaviour of the order parameter in the vicinity of the bifurcation threshold. If low frequencies are absent from the noise spectrum, the order parameter has the same scaling as that of a deterministic bifurcation. In contrast, if the power spectrum does not vanish at zero frequency, the order parameter exhibits anomalous scaling in agreement with recent results for white and harmonic noise.

Our work has allowed us to analyze precisely the role of low frequencies of the noise in a first order random dynamical system. The resummation technique we have used is fairly general and we believe that the results we have derived remain valid for systems of higher order in the vicinity of a stochastic Hopf bifurcation.

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